Planck's Quantum Theory of Black body radiation

Assumptions / postulates

Planck derived an expression for the energy distribution of a black body with the following assumptions –

1. A black body radiator contains oscillators, which are capable of vibrating at all possible frequencies.
2. The energy is radiated in the form of quanta ie in integral multiples of $h\nu$.
3. The frequency of radiation emitted by an oscillator is same as that of the frequency of its vibration.

To derive Planck's Radiation law –

Consider black body radiation inside a cavity. The radiation may be treated as a collection of harmonic oscillators of different frequencies

$u(\nu)d\nu$ is the energy per unit volume of radiation whose frequencies lies between ν and $\nu + d\nu$.

$$u(\nu)d\nu = g(\nu)d\nu \times \langle E \rangle \quad \text{—} (1)$$

where,

$g(\nu)d\nu$ is numbers of oscillators between ν and $\nu + d\nu$ per unit volume

$\langle E \rangle$ is average energy of each oscillator

$$g(\nu)d\nu = \frac{8\pi\nu^2}{c^3}d\nu \quad \text{—} (2)$$

To find $\langle E \rangle$:

Consider N oscillators of frequency ν in thermal equilibrium. Let the total energy be E_{total}.

$$\therefore \langle E \rangle = \frac{E_{total}}{N}. \quad \text{—} (3)$$

According to Maxwell - Boltzmann statistics, the number of oscillators N_r in energy state

rE $(E = h\nu)$ is given by —

$$N_r = N_0 e^{-rE/k_BT}$$

$N_0 \rightarrow$ no of oscillators in ground state

$$\therefore \quad N_0 = N_0$$
$$N_1 = N_0 e^{-E/k_BT} = N_0 x$$
$$N_2 = N_0 e^{-2E/k_BT} = N_0 x^2$$
$$N_3 = N_0 e^{-3E/k_BT} = N_0 x^3$$

here x is e^{-E/k_BT}

$$N = N_0 + N_1 + N_2 + N_3 + \cdots$$
$$= N_0 + N_0 x + N_0 x^2 + N_0 x^3 + \cdots$$
$$= N_0 (1 + x + x^2 + x^3 + \cdots)$$
$$= N_0 \frac{1}{(1-x)}$$

[$\because$ series $1 + x + x^2 + \cdots$ converges to $\frac{1}{1-x}$ when $|x| < 1$]

E_g

$$E_{total} = E_0 N_0 + E_1 N_1 + E_2 N_2 + E_3 N_3 + \cdots$$
$$= 0 \cdot N_0 + 1E \cdot N_0 x + 2E \cdot N_0 x^2 + 3E \cdot N_0 x^3 + \cdots$$
$$= E N_0 x (1 + 2x + 3x^2 + \cdots)$$
$$= E N_0 x \frac{1}{(1-x)^2}$$

Using (3) we get

$$\langle E \rangle = \frac{\dfrac{E \cancel{N_0} x}{(1-x)^{\cancel{2}}}}{\dfrac{\cancel{N_0}}{\cancel{(1-x)}}}$$

or $$\langle E \rangle = \frac{Ex}{1-x} = \frac{E\cancel{x}}{\cancel{x}\left(\frac{1}{x}-1\right)}$$

or $$\langle E \rangle = \frac{E}{e^{E/k_BT}-1} = \frac{h\nu}{e^{h\nu/k_BT}-1}$$

substituting $\langle E \rangle$ and $g(\nu)d\nu$ in (1) we get

$$\boxed{u(\nu)d\nu = \frac{8\pi\nu^2}{c^3}d\nu \times \frac{h\nu}{e^{h\nu/k_BT}-1}}$$

In terms of wavelength

$$\nu = \frac{c}{\lambda} \quad \text{and} \quad |d\nu| = \frac{c}{\lambda^2}d\lambda$$

we get

$$\boxed{u(\lambda)d\lambda = \frac{8\pi hc}{\lambda^5}\frac{d\lambda}{e^{hc/\lambda k_BT}-1}}$$

WIEN'S SPECTRAL DISTRIBUTION FROM PLANCK'S SPETRAL DISTRIBUTION

consider Planck's spectral distribution

$$u(\lambda)d\lambda = \frac{8\pi hc}{\lambda^5} \frac{d\lambda}{e^{hc/\lambda k_B T} - 1} \quad —① .$$

Wien's spectral distribution is applicable only at shorter wavelengths.

when λ is small

$$e^{hc/\lambda k_B T} - 1 \sim e^{hc/\lambda k_B T}$$

$\therefore$ ① becomes

$$u(\lambda)d\lambda = \frac{8\pi hc}{\lambda^5} \frac{d\lambda}{e^{hc/\lambda k_B T}}.$$

say $8\pi hc = A$ & $hc/k_B = B$

$$\therefore \quad u(\lambda)d\lambda = \frac{A}{\lambda^5} e^{-B/\lambda T} d\lambda$$

Above equation is Wien's spectral distribution

Rayleigh-Jeans spectral distribution from Planck's spectral distribution

Consider Planck's spectral distribution

$$u(\lambda)d\lambda = \frac{8\pi hc}{\lambda^5} \frac{d\lambda}{e^{hc/\lambda k_B T} - 1} \qquad \text{— (1)}$$

Rayleigh-Jeans spectral distribution holds good only at longer wavelengths

For small angle $e^{\theta} = 1 + \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \cdots$

when λ is large $\frac{hc}{\lambda k_B T}$ becomes smaller

$$\therefore \quad e^{hc/\lambda k_B T} \sim 1 + \frac{hc}{\lambda k_B T}$$

substituting above in (1) we get

$$u(\lambda)d\lambda = \frac{8\pi hc}{\lambda^5} \frac{1}{\cancel{1} + \frac{hc}{\lambda k_B T} - \cancel{1}} d\lambda$$

or $$u(\lambda)d\lambda = \frac{8\pi \cancel{hc}}{\lambda^{\cancel{5}\,4}} \times \frac{\cancel{\lambda} k_B T}{\cancel{hc}} d\lambda.$$

or $$u(\lambda)d\lambda = \frac{8\pi}{\lambda^4} k_B T \, d\lambda.$$

Hence arrived at Rayleigh-Jeans spectral distribution.