Schrödinger Time dependent wave Equation

→ Schrödinger wave equations deal with matter wave, whose wavelength and wave vector are given by –

$$\lambda = \frac{h}{p} \; ; \quad \vec{k} = \frac{p}{\hbar}$$

→ ψ is wave function of matter waves.

Consider, equation of plane wave moving in +ve x direction, ψ is a function of x & t.

$$\psi = A e^{i(kx - \omega t)} \quad \text{—} (1)$$

[note : eq'n in 3D is $e^{i(\vec{k}\cdot\vec{r} - \omega t)}$

$\vec{k} = k_x \hat{x} + k_y \hat{y} + k_z \hat{z}$

$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$]

k is wave number or x component of wave vector $\vec{k}$.

ω is angular frequency $= 2\pi\nu = 2\pi \frac{E}{h} \quad (\because E = h\nu)$

& $\quad k = \frac{2\pi}{\lambda} = 2\pi \frac{p}{h} = \frac{p}{\hbar}$

$$\boxed{\omega = \frac{E}{\hbar} \quad \& \quad k = \frac{p}{\hbar}.}$$

Now,

differentiating ① wrt x

$$\frac{\partial \psi}{\partial x} = ik A e^{i(kx-\omega t)}$$

Once again differentiating above wrt x, we get

$$\frac{\partial^2 \psi}{\partial x^2} = (ik)^2 A e^{i(kx-\omega t)}$$

or $$\frac{\partial^2 \psi}{\partial x^2} = -k^2 \psi$$

or $$\frac{\partial^2 \psi}{\partial x^2} = -\frac{p^2}{\hbar^2} \psi \quad \text{--- ②}$$

Now,

differentiating ① wrt t, we get

$$\frac{\partial \psi}{\partial t} = -i\omega A e^{i(kx-\omega t)}$$

or $$\frac{\partial \psi}{\partial t} = -i \frac{E}{\hbar} \psi$$

or $$i\hbar \frac{\partial \psi}{\partial t} = E \psi \quad \text{--- ③}$$

Now, $H\psi = E\psi$

and $H = \frac{p^2}{2m} + V$

$$\therefore E\psi = \frac{p^2}{2m}\psi + V(x,t)\psi$$

Using ② & ③ in above equation we get,

$$\boxed{i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x,t)\,\psi}$$

Above eq'n is Schrödinger's time dependent wave equation in 1 D. Second order in space and first order in time.

In 3 D,

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{d y^2} + \frac{\partial^2 \psi}{\partial z^2}\right) + V(x,t)\,\psi$$

or $$\boxed{i\hbar \frac{\partial \psi}{\partial t} = \left(-\frac{\hbar^2}{2m} \nabla^2 + V\right)\psi.}$$

$i\hbar \frac{\partial}{\partial t}$ is Energy operator $\hat{E}$

& $V - \frac{\hbar^2}{2m}\nabla^2$ is Hamiltonian

with $-i\hbar\nabla$ as momentum operator $\hat{p}$