Schrödinger wave equation

Schrödinger equation describes the evolution of the wave function over time.

The Schrödinger equation is a quantum analog to Newton's 2^{nd} law in classical mechanics.

It comes in two forms :

(i) Schrödinger time independent equation
(OR) Time independent Schrödinger wave equation.
(ii) Time dependent Schrödinger wave equation.

TIME INDEPENDENT WAVE EQUATION.

To start with the derivation, we first consider a complex plane wave eq^n given by –

$$\Psi(x,t) = A e^{i(kx-\omega t)} \quad \text{——} (1)$$

For time independent case, potential is independent of time ie $V = V(x)$ alone.

The systems Hamiltonian H is given by

$$H = \underbrace{T}_{\text{kinetic energy}} + \underbrace{V}_{\text{potential energy}}.$$

Hamiltonian is an operator, when operated upon wave function ψ gives systems total energy,

$$H\psi = E\psi.$$

as $T = \frac{p^2}{2m}$ we can write

$$H = \frac{p^2}{2m} + V$$

$$\therefore \left(\frac{p^2}{2m} + V(x)\right)\psi = E\psi. \quad \text{—} (2)$$

To find $p^2\psi(x)$ we differentiate (1). ~~ia~~ w.r.t. x

i.e.

$$\frac{d\psi(x)}{dx} = ikAe^{i(kx-\omega t)}$$

differentiating again we get

$$\frac{d^2\psi}{dx^2} = (ik)^2 Ae^{i(kx-\omega t)}$$

or $$\frac{d^2\psi}{dx^2} = -k^2Ae^{i(kx-\omega t)}.$$

we know that $Ae^{i(kx-\omega t)} = \psi(x)$.

$$\therefore \frac{d^2\psi}{dx^2} = -k^2\psi(x) \quad \text{—} (3)$$

Now wave vector $k = \frac{p}{\hbar}$ $\therefore k^2 = \frac{p^2}{\hbar^2}$

substituting k^2 value in ③ we get.

$$\frac{d^2\psi}{dx^2} = -\frac{p^2}{\hbar^2}\psi(x)$$

or $$-\hbar^2 \frac{d^2\psi}{dx^2} = p^2 \psi(x)$$

$\therefore$ ② becomes

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi.$$

or $$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = \left(E - V(x)\right)\psi.$$

or $$\frac{d^2\psi}{dx^2} = -\frac{2m}{\hbar^2}(E - V(x))\psi$$

or $$\boxed{\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E - V)\psi = 0.} \quad \text{—④}$$

Eq'n ④ represents one dimensional Schrödinger time independent wave equation.

In 3D,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\psi + \frac{2m}{\hbar^2}(E - V)\psi = 0.$$

$$\nabla^2\psi + \frac{2m}{\hbar^2}(E - V)\psi = 0$$

where ∇^2 is Laplacian operator.